More Tales of Two (s)-ities.

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Abstract: *In any complete separable metric space, the Boolean algebra* $(s)/(s_0)$ *of Marczewski sets modulo the Marczewski null sets is* complete*. Using this fact, we show that a simple construction solves two known problems in real analysis: the existence of an* $(s)_0$*-set which is not Lebesgue measurable and does not have the Baire property, and a function which is not* (s)*-measurable, but whose graph is an* $(s)_0$*-set. Using a similar idea, we also present a short proof that the Boolean algebra of universally measurable sets modulo the sets universally of measure zero is* not *complete.*

Let X and Y be complete separable metric spaces. In [S], we constructed by a fairly simple method an (s)-set in $X \times Y$ which does not belong to the σ-algebra generated by products $A \times B$ of (s)-sets $A \subset X$ and $B \subset Y$. The key idea was to apply the fact that $(s)/(s_0)$ is a *complete* Boolean algebra, and to find a collection of (s)-sets whose least upper bound in that algebra has the desired properties. Here we show that this same construction can also be used to give short solutions to other known problems in the theory of (s)-sets; namely, the existence of an (s_0)-set which is not Lebesgue measurable and does not have the Baire property, and a function which is not (s)-measurable but whose graph is an (s)-set. Finally, we show that a slight variant of this construction yields a short proof of the fact that the Boolean algebra of universally measurable sets modulo the universally null sets (as well as the corresponding category algebra) is not complete.

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A set M in the complete separable metric space Z is said to have property (s) if every perfect set in Z has a perfect subset which is either a subset of M or is disjoint from M. The set M has property (s_0) if every perfect set in Z has a perfect subset which is disjoint from M. These notions were introduced by Szpilrajn-Marczewski in [SM], where it is proved, among many other things, that the class of sets having property (s) is a σ-algebra, and the class of sets having propery (s_0) is a σ-ideal. Baldwin and Brown (in [BB]) recently gave a pithy proof of the fact that the Boolean algebra $(s)/(s_0)$ is in fact a *complete* Boolean algebra. Now let us work in the space $X \times Y$, where X and Y are both complete separable metric spaces.

For $x \in X$, let $C_x = \{x\} \times Y$ be the vertical cross-section of $X \times Y$ at x, and for $y \in Y$, let $D_y = X \times \{y\}$ be the horizontal cross-section of $X \times Y$ at y.

The key idea in [S] was simply this; let $\hat{M}$ be the least upper bound in the complete Boolean algebra $(s)/(s_0)$ (in the space $X \times Y$) of the sets C_x, over all $x \in X$. (Of course, $\hat{M}$ is really determined only up to (s_0)-sets; take $\hat{M}$ to be any set in the appropriate equivalence class modulo (s_0).) We have

Proposition 1
(i) $\hat{M}$ is has property (s);
(ii) For all $x \in X$, $C_x \setminus \hat{M}$ has property (s_0);
(iii) For all $y \in Y$, $D_y \cap \hat{M}$ has property (s_0);

This much follows from the definition of least upper bound. We showed in [S] that while $\hat{M}$ is an (s)-set, it does not belong to the σ-algebra generated by the Cartesian products of (s)-sets in X and in Y.

In [W], Walsh constructed a set which has property (s_0) but does not have the Baire property and is not measurable. We show here that, in fact, in the set $\hat{M}$ we can find a set having (and lacking) exactly these properties.

In [ET], Elalaoui-Talibi proves that the graph of every (s)-measurable function is an (s)-set, and remarks that the converse is open in ZFC; in other words, there is no known example of a function which is not (s)-measurable, but whose graph is an (s)-set; in [SM] Szpilrajn-Marczewski constructed an example using the continuum hypothesis. We shall show here that any function whose graph is a subset of $\hat{M}$ is an (s_0)-set, and that it is easy to make such a function fail to be (s)-meausrable.

A set M in Z is said to be *universally measurable* if M is measurable with respect to every complete finite Borel measure on Z, and *universally of measure zero* if M has measure zero with respect to every continuous complete finite Borel measure. Similarly, M is said to have the *Baire property* *in the restricted sense* if $M \cap P$ has the Baire property in P for every perfect set P in Z, and be *always of first category* if $M \cap P$ is of the first category for all perfect sets P in Z.

The σ-algebra of universally measurable sets is denoted $\mathcal{U}$, and the σ-ideal of sets universally of measure zero is denoted $\mathcal{U}_0$. The σ-algebra of sets having the Baire property in the restricted sense is denoted $\mathcal{B}_r$, and the σ-ideal of always first category sets is denoted AFC. In [B], Baldwin uses a deep result of Miller to show that that the Boolean algebras $\mathcal{U}/\mathcal{U}_0$ and $\mathcal{B}_w/AFC$ are *not* complete. In section 2.3, we give a short proof using a construction in the spirit of $\hat{M}$.

2 Applications.

2.1 There exists an (s_0)-set which is not Lebesgue measurable and does not have the Baire property.

Let $X = Y =$ the closed interval $[0, 1]$ on the real line, and let $\hat{M}$ be the set from section 1. The set $\hat{M}$ is not Lebesgue measurable. Indeed, since by Proposition 1 no horizontal cross section $D_y \cap \hat{M}$ contains a perfect set, each *measurable* horizontal cross section $D_y \cap \hat{M}$ must have measure zero. Thus Fubini's theorem implies that if $\hat{M}$ is measurable, its measure must be 0. Similarly, each *measurable* vertical cross-section $C_x \cap \hat{M}$ must have measure 1, so if $\hat{M}$ is measurable, its measure must be 1. Thus $\hat{M}$ is an example of an (s)-set which is not measurable. Using Fubini's theorem once again, there exists $x_0 \in [0, 1]$ such that $\{y \in [0, 1] : (x_0, y) \in \hat{M}\}$ is not measurable.

Now we can repeat the whole argument using Baire category in place of measure to show that $\hat{M}$ also fails to have the Baire property, and we obtain in a similar way $x_1 \in [0, 1]$ such that $\{y \in [0, 1] : (x_1, y) \in \hat{M}\}$ does not have the Baire property.

Thus $\hat{M}$ is an (s)-set which is not Lebesgue measurable and does not have the Baire property. To obtain an (s_0)-set which is not Lebesgue measurable and does not have the Baire property, consider $\hat{M}'$, the subset of the space $[0, 2]$ given by $\hat{M}' = \{y \in [0, 2] : (x_0, y) \notin \hat{M}$ and $(x_1, y - 1) \notin \hat{M}\}$; $\hat{M}'$ is the required set.

Of course, there is nothing special about Lebesgue measure and the real line; a similar construction works for any finite continuous Borel measure on any uncountable separable complete metric space.

2.2 There exists a function $f : X \to Y$ which is not (s)-measurable, but whose graph is an (s_0)-set.

Let X and Y be any uncountable complete separable metric spaces, and again let $\hat{M}$ be as in section 1. We show that the graph of any function $g : X \to Y$ which is a subset of $\hat{M}$ must be an (s_0)-set. Indeed, let $P \subset X \times Y$ be any perfect set. If $P \cap C_x$ is uncountable for some $x \in X$, then $P \cap C_x \setminus \{(x, g(x)\}$ is an uncountable Borel set, and so has a perfect subset which is clearly disjoint from the graph of g. If, on the other hand, $P \cap C_x$ is countable for all x, then since $P \cap \hat{M}$ is the least upper bound in the Boolean algebra $(s)/(s_0)$ of the sets $P \cap C_x$ over all $x \in X$, $P \cap \hat{M}$ is an (s_0)-set, and therefore $P \setminus \hat{M}$ has a perfect subset, which is also clearly disjoint from the graph of g.

Now let $Y_1, Y_2 \subset Y$ be disjoint uncountable Borel sets, and let $Z \subset X$ be any set which does not have property (s). Since each cross-section of $\hat{M} \cap C_x$ is the complement in C_x of an (s_0)-set, for all $x \in X$ we can choose $g_1(x) \in Y_1$ and $g_2(x) \in Y_2$ so that $(x, g_1(x)), (x, g_2(x)) \in \hat{M}$. Define the function $f : X \to Y$ by putting $f(x) = g_1(x)$ for $x \in Z$, $f(x) = g_2(x)$ for $x \in X \setminus Z$. Since $f^{-1}(Y_1) = Z$, f is not (s)-measurable, but the graph of f has property (s_0).

2.3 The Boolean algebras $\mathcal{U}/\mathcal{U}_0$ and $\mathcal{B}_r/AFC$ are not complete.

Once again, let X and Y be uncountable complete separable metric spaces. Let $\mathcal{U}$ ($\mathcal{U}_0$) be the class of universally measurable (universally measure zero) sets in $X \times Y$. We show that $\mathcal{U}/\mathcal{U}_0$ is not complete; the proof that $\mathcal{B}_r/AFC$ is not complete is analogous.

We shall show that in fact the class of sets C_x (over all $x \in X$) has no least upper bound. Indeed, suppose for the purpose of contradiction that $\hat{N}$ is this least upper bound.

Let μ, ν be nonzero continuous Borel probability measures on X, Y respectively. For each $x \in X$, consider the nonzero continuous Borel probability measure ν_x on $X \times Y$ given by $\nu_x(B) = \nu(\{y \in Y : (x, y) \in B\})$. Since by construction $\hat{N} \geq C_x$, $\nu_x(C_x \cap N) = \nu_x(C_x) = 1$. By Fubini's theorem, $(\mu \times \nu)(\hat{N}) = 1$.

Similarly, for each $y \in Y$, define the Borel measure μ_y on $X \times Y$ by $\mu_y(B) = \mu(\{x \in X : (x, y) \in B\})$. Since $C_x \cap D_y = \{(x, y)\} \in \mathcal{U}_0$, for all x, y, and $\hat{N} \cap D_y$ is the least upper bound in the Boolean algebra $\mathcal{U}/\mathcal{U}_0$ of the sets $C_x \cap D_y$ over all $x \in X$, it follows that $\hat{N} \cap D_y \in \mathcal{U}_0$ for all y. In particular, $\nu_y(\hat{N} \cap D_y) = 0$ for all $y \in Y$, so again by Fubini, $(\mu \times \nu)(\hat{N}) = 0$. This contradiction completes the proof.

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